

Calculus III: Optional Coursework 2
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Note: Although this course work is optional, those who decide not to do it would be well-advised to make sure they understand the solutions, since related questions may appear in the final exam.

The deadline is 4PM on Wednesday Dec 12th. Hand-in is to the green box on the ground floor of the Maths building.

1. The function $f(x)$ is periodic with period 3ℓ and is defined by [25]

$$f(x) = \begin{cases} 0 & \text{if } 0 < x < \ell \\ a & \text{if } \ell < x < 3\ell. \end{cases}$$

Find the Fourier series for this function.

To what does it converge at $x = 0$ and $x = \ell$?

Use it to show that

$$\sum_1^{\infty} \frac{\sin(2k\pi/3)}{k} = \frac{\pi}{6}.$$

Answer

The series will be $\frac{1}{2}a_0 + \sum_1^{\infty} (a_n \cos(2n\pi x/3\ell) + b_n \sin(2n\pi x/3\ell))$ and

$$a_m = \frac{1}{P} \int_0^{2P} f \cos(m\pi x/P) dx$$

where $2P = 3\ell$ is the period. (In the lectures I called it 2ℓ so some students may be confused by the re-use of ℓ here.) Similarly for b_m . The integrals only have a non-zero contribution where $f \neq 0$ so (using $y = 2\pi x/3\ell$)

$$\begin{aligned} a_0 &= \frac{2}{3\ell} \int_{\ell}^{3\ell} a dx = \frac{4a}{3}, \\ a_m &= \frac{2}{3\ell} \int_{\ell}^{3\ell} a \cos\left(\frac{2m\pi x}{3\ell}\right) dx = \frac{1}{\pi} \int_{2\pi/3}^{2\pi} a \cos my dy \\ &= \frac{a}{\pi} \left[\frac{\sin my}{m} \right]_{2\pi/3}^{2\pi} = -\frac{a \sin(2m\pi/3)}{m\pi} = -\frac{2a}{m\pi} \sin \frac{n\pi}{3} \cos \frac{n\pi}{3} \\ b_m &= \frac{2}{3\ell} \int_{\ell}^{3\ell} a \sin\left(\frac{2m\pi x}{3\ell}\right) dx = \frac{1}{\pi} \int_{2\pi/3}^{2\pi} a \sin my dy \\ &= \frac{a}{\pi} \left[-\frac{\cos my}{m} \right]_{2\pi/3}^{2\pi} = -\frac{a}{m\pi} \left(1 - \cos \frac{2m\pi}{3} \right) = -\frac{2a}{m\pi} \sin^2 \frac{n\pi}{3} \end{aligned}$$

Thus the whole series is

$$\frac{2a}{3} - \frac{2a}{\pi} \sum_1^{\infty} \frac{1}{n} \sin \frac{n\pi}{3} \left(\cos \frac{n\pi}{3} \cos \frac{2n\pi x}{3\ell} + \sin \frac{n\pi}{3} \sin \frac{2n\pi x}{3\ell} \right)$$

This can also be written as

$$\frac{2a}{3} - \frac{2a}{\pi} \sum_1^{\infty} \frac{1}{n} \sin \frac{n\pi}{3} \cos \frac{n\pi}{3\ell} (\ell - 2x)$$

At $x = 0$ and $x = \ell$ this must converge to $\frac{1}{2}a$ (these being discontinuities).

Evaluating at $x = 0$ we get

$$\begin{aligned} \frac{1}{2}a &= \frac{2a}{3} - \frac{2a}{\pi} \sum_1^{\infty} \frac{1}{n} \sin \frac{n\pi}{3} \cos \frac{n\pi}{3} \\ \frac{2}{3} - \frac{1}{2} &= \frac{1}{\pi} \sum_1^{\infty} \frac{1}{n} \sin \frac{2n\pi}{3} \\ \frac{\pi}{6} &= \sum_1^{\infty} \frac{1}{n} \sin \frac{2n\pi}{3} \end{aligned}$$

Marks: 5 for (at least one) correct formula for the coefficients, 2 for a_0 , 4 for each of a_m and b_m , 2 for putting it all together, 3 for the values at the jumps, 5 for the last part.

2. Find the solution of

[10]

$$\nabla^2 \Phi \equiv \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$$

in the domain D : $0 \leq x \leq a$, $0 \leq y \leq b$, given boundary conditions $\Phi = 0$ on $x = 0$, on $y = 0$ and on $y = b$, and $\Phi = c \sin(k\pi y/b) \cos(k\pi y/b)$ (for some integer k and constant c) on $x = a$.

Answer

The boundary condition on $x = a$ can be written $\Phi = \frac{1}{2}c \sin(2k\pi y/b)$. Inspecting the separable solutions of Laplace's equation we see that we need only to take the solution of the form $A \sinh(2k\pi x/b) \sin(2k\pi y/b)$ and then matching the values on $y = b$ gives $A = c/2 \sinh(2k\pi a/b)$ whence

$$\Phi = \frac{c \sinh(2k\pi x/b) \sin(2k\pi y/b)}{2 \sinh(2k\pi a/b)}.$$

Marks: 3 for writing the BC as a Fourier series (of one term), 4 for realizing only one separable solution is needed, 3 for correct coefficients etc.

Note: The bookwork part of exam questions on this topic, which you are not asked to do as part of the coursework, might say:

Consider Laplace's equation $\nabla^2 V = 0$ in the case where V is a function of Cartesian coordinates x and y only. Find all possible separable solutions of the form $V(x, y) = X(x)Y(y)$. Describe the form of the general solution.

3. A function $F(x, y)$ satisfies Laplace's equation in the square $x, y \in [0, 1]$ with boundary values as follows: [35]

$$F = 0 \text{ on } y = 0$$

$$F = y \text{ on } x = 0 \text{ and } x = 1$$

$$F = \cos 2\pi x \text{ on } y = 1.$$

This $F(x, y)$ can be split into two functions

$$F(x, y) = f(x, y) + g(x, y)$$

such that at the corners $f(x, y) = 0$ and $g(x, y) = F(x, y)$.

- (a) Find $g(x, y)$.
- (b) Find $f(x, y)$ on the sides.
- (c) Find the Fourier sine series for $f(x, 1)$.
- (d) Hence find $f(x, y)$ and $F(x, y)$.

Answer

(a) Start with $g = \alpha + \beta x + \gamma y + \delta xy$. The values at the corners are $\Phi(0, 0) = \Phi(1, 0) = 0$, $\Phi(0, 1) = \Phi(1, 1) = 1$. Equating these to g at those points we see that $\alpha = \beta = 0$, $\gamma = 1$, $\delta = 0$, so $g = y$.

(b) $f = 0$ on the sides $y = 0$, $x = 0$ and $x = 1$. $f = \cos(2\pi x) - 1$ on $y = 1$.

(c) To get the sine series we need to evaluate $b_m = 2 \int_0^1 f \sin(m\pi x) dx$.

$$\begin{aligned} b_m &= 2 \int_0^1 (\cos(2\pi x) - 1) \sin(m\pi x) dx \\ &= 2 \left[\frac{\cos(m\pi x)}{m\pi} \right]_0^1 + \int_0^1 (\sin((m-2)\pi x) + \sin((m+2)\pi x)) dx \\ &= 2 \frac{(-1)^m - 1}{m\pi} - \left[\frac{\cos((m-2)\pi x)}{(m-2)\pi} + \frac{\cos((m+2)\pi x)}{(m+2)\pi} \right]_0^1 \quad (\text{if } m \neq 2) \\ &= 2 \frac{(-1)^m - 1}{m\pi} - \left[\frac{(-1)^{m-2} - 1}{(m-2)\pi} + \frac{(-1)^{m+2} - 1}{(m+2)\pi} \right]_0^1 \\ &= 2 \frac{(-1)^m - 1}{\pi} \left(\frac{1}{m} - \frac{m}{m^2 - 4} \right) = \frac{8(1 - (-1)^m)}{m(m^2 - 4)\pi} \end{aligned}$$

which is zero for even m and $16/(m(m^2 - 4)\pi)$ for odd m .

The special case $m = 2$ gives

$$\begin{aligned} b_2 &= 2 \int_0^1 (\cos(2\pi x) - 1) \sin(2\pi x) dx = 2 \left[\frac{\cos(2\pi x)}{2\pi} \right]_0^1 + \int_0^1 \sin(4\pi x) dx \\ &= - \left[\frac{\cos(4\pi x)}{4\pi} \right]_0^1 = 0. \end{aligned}$$

Thus, writing $n = 2k + 1$ for odd n , the Fourier series is

$$\sum_{k=1}^{\infty} \frac{16 \sin((2k+1)\pi x)}{(2k+1)((2k+1)^2 - 4)\pi}$$

(Note to students: the integrations here are very similar to the one you did for the square wave in exercises 8, and the one for $\cos x$ on $[0, \pi]$ that was in last year's exam and was given as an additional exercise in the lecture notes, section 7.1.)

(d) Matching the series found for the values at $y = 1$ to the solution

$\Phi = \sum_1^\infty D_n \sinh(n\pi y) \sin(n\pi x)$ (special case of the one discussed in lectures) we find

$$D_{2k} = 0, \quad D_{2k+1} = \frac{16}{(2k+1)((2k+1)^2 - 4)\pi \sinh((2k+1)\pi)} .$$

Thus the whole solution is

$$F = y + \sum_{k=1}^{\infty} \frac{16 \sinh((2k+1)\pi y) \sin((2k+1)\pi x)}{(2k+1)((2k+1)^2 - 4)\pi \sinh((2k+1)\pi)}$$

Marks:

(a) correct method 2, correct equations 3, solving and final answer 3 (Total 8 for this part)

(b) 2

(c) correct formula for sine series in this case 3. Integration of the 1, 3 marks, and of the cos, 8 marks. 1 extra if the results are recombined neatly (no further penalty for correct but messier forms). 2 for dealing with $m = 2$. 2 for final series (Total for this part 19.)

(d) correct method 3, final answer 3 (allow follow through from their answer to (c))

4. Consider the region $\mathcal{D}$ defined in cylindrical polar coordinates by $1 \leq \rho \leq 2$. Find the solution Φ of Laplace's equation, independent of z , which satisfies the following boundary conditions: (i) $\Phi = \cos 2\phi + \cos \phi$ when $\rho = 1$ and (ii) $\Phi = 3 \cos \phi$ when $\rho = 2$. [15]

Answer

Since only $\cos 2\phi$ and $\cos \phi$ occur in the boundary conditions, we can guess that

$$\Phi = (A\rho + B/\rho) \cos \phi + (C\rho^2 + D/\rho^2) \cos 2\phi .$$

To match at $\rho = 2$ we need $2A + B/2 = 3$ and $4C + D/4 = 0$ and to match at $\rho = 1$ we need $A + B = 1$ and $C + D = 1$. Solving these conditions we obtain $(A, B, C, D) = (\frac{5}{3}, -\frac{2}{3}, -\frac{1}{15}, \frac{16}{15})$ and hence

$$\Phi = \frac{1}{3} \left(5\rho - \frac{2}{\rho} \right) \cos \phi + \frac{1}{15} \left(\frac{16}{\rho^2} - \rho^2 \right) \cos 2\phi .$$

Marks: Correct and suitable form for Φ 4 (no penalty for extra terms eventually not needed). Equations from BCs 3. Solving 6. Final form 2. Allow part marks and follow through from arithmetic mistakes.

5. A potential Φ satisfies Laplace's equation in the region outside a sphere of radius a . Far from the sphere (i.e. "at infinity") $\mathbf{F} = \nabla \Phi$ is a constant vector. Taking spherical

polar coordinates (r, θ, ϕ) , with the polar-axis along the direction of $\mathbf{F}$ at infinity, Φ on the surface $r = a$ is given by $\Phi = 1 + \cos 2\theta$. Find Φ . [You may assume the solution is axisymmetric.] [1]5

Answer

At infinity we need only a term such that

$$\nabla\Phi = \mathbf{F} = F\mathbf{k} = F \cos \theta \mathbf{e}_r - F \sin \theta \mathbf{e}_\theta .$$

Hence at infinity we need just $\Phi_0 = Fr \cos \theta = Fr P_1(\cos \theta)$. The condition on the surface a is

$$\phi = 1 + \cos 2\theta = 2 \cos^2 \theta = \frac{4}{3} P_2(\cos \theta) + \frac{2}{3} .$$

To match all these terms we will need something of the form

$$\Phi = \frac{B_0}{r} + \left(Fr + \frac{B_1}{r^2} \right) P_1(\cos \theta) + \frac{B_2}{r^3} P_2(\cos \theta)$$

and it is then easy to obtain $B_0 = \frac{2a}{3}$, $B_1 = -Fa^3$, $B_2 = \frac{4a^3}{3}$ so

$$\Phi = \frac{2a}{3r} + \left(Fr - \frac{Fa^3}{r^2} \right) P_1(\cos \theta) + \frac{4a^3}{3r^3} P_2(\cos \theta)$$

Marks: (This was intentionally very similar to the example in the lectures, which is why the marks are not too generous.) Working out BC at infinity and giving that part of Φ 4. Expressing BCs on sphere in P_ℓ 3. Some correct form of general Φ 3. Matching on the sphere 3. Putting it together 2.

Do not penalise people who take the tail to be $\sum_{n=2}^{\infty} \frac{B_n}{r^{n+1}} P_n(\cos \theta)$ and then argue that $B_n = 0$ for $n > 2$.